Indian Institute of Information Technology Allahabad
Discrete Mathematical Structures (DMS)
C3 Review Test: Tentative Marking Scheme

Program: B.Tech. 2nd Semester (IT) Full Marks: 50
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1. Check whether the following statements are true or false. Give proper justification. [12]

   (a) If H is a subgroup of an abelian group G, then H is a normal subgroup of G.

   Solution: True. Let $x \in G, h \in H$, we get

   $$\begin{aligned} xhx^{-1} &= xx^{-1}h \ \text{ (as } G \text{ is abelian)} \\ &= eh = h \in H. \end{aligned}$$ [2]

   (b) Let $(\mathbb{Z}, +)$ be a group. The number of distinct left cosets of $6\mathbb{Z}$ in $\mathbb{Z}$ is 7.

   Solution: False. These are exactly 6 left cosets: $0 + 6\mathbb{Z}, 1 + 6\mathbb{Z}, 2 + 6\mathbb{Z}, 3 + 6\mathbb{Z}, 4 + 6\mathbb{Z}, 5 + 6\mathbb{Z}$. [2]

   (c) Let $A, B \subset \mathbb{R}$ be two countable sets. Then the sum $A + B := \{a + b : a \in A, b \in B\}$ is countable.

   Solution: True. $A + B = \cup_{a\in A}(a + B)$. [1]

   As $a + B$ is equivalent to B, so $a + B$ is countable.

   Also, the countable union of the countable set is countable, therefore $A + B = \cup_{a\in A}(a + B)$ is countable. [1]

   (d) Every connected graph with n vertices has at least $n - 1$ edges.

   Solution: True. Since a connected graph has a tree as a subgraph. [2]

   (e) Every cyclic graph C_n $(n \geq 3)$ is a bipartite graph.

   Solution: False. C_n is bipartite if and only if n is even. [2]

(f) There exists a self-complimentary graph with 6 vertices.

Solution: False. Since G is isomorphic to its complement $\overline{G}$, they have the same number of edges, i.e., $E(G) = E(\overline{G})$.

Note that $E(G) + E(\overline{G}) = \frac{n(n-1)}{2}$.

Then $E(G) = \frac{n(n-1)}{4}$. This is only possible if n or $n-1$ is divisible by 4. For instance, a 6-vertex graph cannot be self-complementary. [2]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Write the discontinuity of f at $x_0 \in \mathbb{R}$ in terms of predicates and quantifiers. [3]

Solution: f is continuous at x_0 can be written as

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon).$$

Thus, f is discontinuous at x_0 is expressed as

$$\exists \ \epsilon > 0 \ \forall \delta > 0 \ \exists x \ \neg(|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon),$$ [2]

$$\exists \ \epsilon > 0 \ \forall \delta > 0 \ \exists x \ (|x - x_0| < \delta \wedge |f(x) - f(x_0)| \geq \epsilon).$$ [1]

3. Show that every composite number can be written (factorized) as the product of primes and this factorization is unique, apart from the order in which the prime factors occur. [5]

Solution: We prove it using a strong form of mathematical induction. For $n \in \mathbb{N}$, $P(n) : n$ is prime or n is a product of prime. Clearly, 2 is prime, so $P(2)$ is true. Assume $P(m)$ is true, for all m, where $2 \leq m \leq k$. Now to show $P(k+1)$ is true. If $k+1$ is prime, then we are done. If not, there exists $a, b \in \mathbb{N}$, such that $k+1 = ab$. Clearly, $a, b < k+1$. Therefore, using induction hypothesis $a = p_1 p_2 \ldots p_r$, $b = q_1 q_2 \ldots q_s$, where each p_j, q_j are primes. Then $k+1 = p_1 p_2 \ldots p_r q_1 q_2 \ldots q_s$ is a product of primes. Thus, $P(k+1)$ is true. [3]

Uniqueness: Let n be the least positive integer that has two distinct prime factorizations, that is, $n = q_1q_2 \dots q_k = p_1p_2 \dots p_l$, where each p_j, q_j are prime. Now, we see that q_1 divides $p_1p_2 \dots p_l$, so there exists $1 \le i \le l$ such that q_1 divides p_i. WLOG, q_1 divides p_1, say. It is possible only if $q_1 = p_1$, as p_1 and q_1 are primes. Thus, we are left with $q_2q_3 \dots q_k = p_2p_3 \dots p_l$. It is two distinct prime factorizations of some positive integer strictly less than n, which contradicts that n be the least positive integer with two distinct prime factorizations. [2]

4. Let H be a subgroup of a group G. Show that the relation R on G defined as xRy iff $xy^{-1} \in H$ is an equivalence relation. [3]

Solution:

- Reflexive: $\forall x \in G, xx^{-1} = e \in H$. Thus, xRx. [1]
- Symmetric: Let $x, y \in G$ such that xRy. This gives $xy^{-1} \in H$. Since H is a subgroup of G, so $(xy^{-1})^{-1} = (y^{-1})^{-1}x^{-1} = yx^{-1} \in H$. Thus, yRx. [1]
- Transitive: Let $x, y, z \in G$ such that xRy and yRz. This gives $xy^{-1}, yz^{-1} \in H$. Since H is a subgroup of G, so $(xy^{-1})(yz^{-1}) = x(y^{-1}y)z^{-1} = xz^{-1} \in H$. This gives xRz. [1]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that the set of discontinuity of f, that is, $D = \{x \in \mathbb{R} : \ f \text{ is discontinuous at } x\}$, is countable. [6]

Solution: If $D = \emptyset$, we are done.

If not, assume $t \in D$. Since f is increasing function and discontinuous at t, therefore $\lim_{x\to t^-} f(x) < \lim_{x\to t^+} f(x)$. [1]

Then there exists $r_t \in \mathbb{Q}$ such that $\lim_{x\to t^-} f(x) < r_t < \lim_{x\to t^+} f(x)$. [1]

Let $s \in D,\ s \neq t$. WLOG assume $t < s$, then $\lim_{x\to t^+} f(x) \le \lim_{x\to s^-} f(x)$. [1]

Therefore, $r_t < \lim_{x\to t^+} f(x) \le \lim_{x\to s^-} f(x) < r_s < \lim_{x\to s^+} f(x)$, where $r_s \in \mathbb{Q}$ such that $\lim_{x\to s^-} f(x) < r_s < \lim_{x\to s^+} f(x)$. [1]

Define a map $\Phi : D \to \mathbb{Q}$ as $\Phi(t) = r_t$. Then Φ is one-one. Thus, D is countable. [2]

Alternate Solution: [6]

Let $x_0 \in [a, b]$ be the point of discontinuity of f. Since f is increasing function and discontinuous at x_0, therefore $\lim_{x\to x_0^-} f(x) < \lim_{x\to x_0^+} f(x)$. Consider set $D_n = \{r \in [a, b] : \lim_{x\to r^+} f(x) - \lim_{x\to r^-} f(x) > \frac{1}{n}\}$. Now to show D_n is finite. If not, i.e., D_n is an infinite set. Let $x_1, x_2, x_3, \cdots \in D_n$. Then

$$\sum_{n=1}^{\infty}(\lim_{x\to x_n^+} f(x) - \lim_{x\to x_n^-} f(x)) > \sum \frac{1}{n}.$$

Since $\sum \frac{1}{n}$ is divergent so $\sum_{n=1}^{\infty}(\lim_{x\to x_n^+} f(x) - \lim_{x\to x_n^-} f(x))$ is divergent, but $\sum_{n=1}^{\infty}(\lim_{x\to x_n^+} f(x) - \lim_{x\to x_n^-} f(x)) \leq f(b) - f(a)$. This gives a contraction. Therefore, the set $F = \{x \in [a,b] : f \text{ is discontinuous at } x\} = \bigcup_{n=1}^{\infty} D_n$ is countable, being the countable union of a finite set. Now $\mathbb{R} = \bigcup_{a\in\mathbb{Z}}[a, a+1]$, which verifies the claim.

6. Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, with the initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$. [7]

Solution: The characteristic polynomial of the recurrence relation is given as $r^3 + 3r^2 + 3r + 1$, which is $(r+1)^3$. [1]

The characteristic root is $r = -1$. [1]

So the solution to the recurrence relation is of the form

$$a_n = (\alpha + \beta n + \gamma n^2)(-1)^n, \quad [2]$$

where α, β, γ are recognized using initial condition. Put $a_0 = 1$, we get $\alpha = 1$. Again using $a_1 = -2, a_2 = -1$, we get $\beta = 3, \gamma = -2$. [1+1+1]

Thus, the solution to the recurrence relation is

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

7. How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $2 \leq x_1 \leq 7, 0 \leq x_2 \leq 5$ and $6 \leq x_3$ are integers? [3]

Solution: The generating function is given as [1]

$$\begin{aligned}\phi(x) &= (x^2+x^3+x^4+x^5+x^6+x^7)(1+x+x^2+x^3+x^4+x^5)(x^6+x^7+x^8+\ldots)\\ &= x^8(1+x+x^2+x^3+x^4+x^5)^2(1+x^1+x^2+x^3+\ldots)\\ &= x^8\Big(\frac{1-x^6}{1-x}\Big)^2\Big(\frac{1}{1-x}\Big)\\ &= x^8(1-x^6)^2(1-x)^{-3}\\ &= x^8(1+x^{12}-2x^6)\sum_{k=0}^{\infty}\binom{-3}{k}(-1)^k x^k \qquad [1]\end{aligned}$$

The coefficient of x^{10} is obtained as $\binom{-3}{2}(-1)^2 = 6$. Thus, the required number of solutions for the given equation is 6. [1]

8. Let $G=(V,E)$ be a graph with the vertex set V and edge set E as follows: [11]

$$V=\{0,1,\ldots,9\} \text{ and } E=\{01,02,03,14,15,26,27,38,39,47,48,56,59,68,79\}.$$

(a) Write the vertex connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$ of G.

Solution: The vertex connectivity $\kappa(G)=3$. [1]

The edge connectivity $\lambda(G)=3$. [1]

(b) Write the eccentricity of vertices 0 and 2.

Solution: $e(0)=e(2)=2$. [1/2+1/2]

(c) Write the radius, circumference, center, grith, and clique number of G.

Solution: The radius of G is 2. [1]

The circumference of G is 9. [1]

The center of G is V. [1]

The girth of G is 5. [1]

The clique number of G is 2. [1]

(d) Is G planar? Justify your answer.

Solution: G is not planar.

$\because$ girth of G is 5, $e \geq \frac{5f}{2} \iff f \leq \frac{2e}{5}$. [1]

Now, if G is planar, then $v-e+f=2 \iff 2-v+e=f$.

This gives $7 = 2 - v + e = f \leq \frac{2e}{5} = 6$. A contradiction. [2]

Alternatively: For $V' = \{0, 1, 2, 3, 6, 8, 9\}$, the spanning subgraph $G =< V' >$ is homeomorphic to $K_{3,3}$, which is non-planar. [3]